Extending states on finite concrete logics

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Abstract

In this note we collect several observations on state extensions. They may be instrumental to anyone who pursues the theory of quantum logics. In particular, we find out when extensions (resp. signed extensions) exist in the "concrete" concrete logic of all even-element subsets of an even-element set (Th 2.3 and Th 2.9). We also mildly add to the study of difference-closed logics as initiated in [10] by finding an extension theorem for subadditive states. Our results suplement the research carried on in [2], [4], [5], [8], [7], [10], [13], [14], [15], [16], and [17].

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1 Introduction

The question of extending states on quantum logics is sometimes surprisingly combinatorially involved. In spite of the progress made by the authors referred to in the abstract above, several questions remain open (see e.g. [10] and [13]). It therefore seems helpful to have the situation clarified in the "testing" case of the logic of even-number-element subsets of a set. This is what we intend to do in this note. Our results may partially overlap with the results of the previous effort but we are not aware of them being explicitly formulated elsewhere.

2 Notions and results

We shall exclusively deal with finite concrete (= set representable) quantum logics and states (= probability measures) on them as defined below. Standardly, for a set X we stand $\exp(X)$ for the (Boolean) power algebra of X.

Definition 2.1 *Let X be a finite set. A collection $\Delta \subseteq \exp X$ is said to be a concrete quantum logic (abbr. a* logic*) if the following conditions are satisfied:*

1. *$X \in \Delta$;*

2. *if $A \in \Delta$, then $A^c \in \Delta$ (A^c denotes the complement of A in X);*

3. *if $A, B \in \Delta$ and $A \cap B = \emptyset$, then $A \cup B \in \Delta$.*

Let $m : \Delta \to \mathbb{R}$ be a mapping from a logic into the set of real numbers. We say that m is a signed measure *on Δ if $m(A \cup B) = m(A) + m(B)$ provided that $A, B \in \Delta$ and $A \cap B = \emptyset$. A signed measure $m : \Delta \to \mathbb{R}$ is said to be a* state *if $m(A) \geq 0$ for any $A \in \Delta$ and $m(X) = 1$.*

The chief question we ask here reads as follows: Having given a (concrete) logic Δ on a set X and having given a state s on Δ, when can we extend s as a state (resp. as a signed measure) over the entire algebra $\exp X$? Expressed more formally, given a state s on Δ, when can we find a state t (resp., a signed measure t) on the Boolean algebra $\exp X$ such that t restricted to Δ equals to s? Let us observe first that certainly not always. In fact, we even do not have the "weak" extensions (= extensions of states to signed measures) of two-valued states at our disposal as the following simple example shows.

Example 2.2 *Let $X = \{1, 2, \ldots, 6\}$, and let Δ be the smallest concrete logic on X containing the following four sets:*

$$A = \{1, 2, 3\}, \qquad B = \{2, 3, 4\}, \qquad C = \{3, 4, 5\}, \qquad D = \{1, 3, 5\}.$$

Obviously, $\Delta = \{\emptyset, A, A^c, B, B^c, C, C^c, D, D^c, X\}$ (for the reader acquainted with the theory of orthomodular lattices, Δ is a representation of the orthomodular lattice MO_4). Let us take the (two-valued) state $s : \Delta \to \{0, 1\}$ defined as follows:

$$s(A) = 0, \quad s(B) = 1, \quad s(C) = 0, \quad s(D) = 1.$$

We claim that this state s cannot be extended over $\exp X$ as a signed measure. Indeed, suppose that $m : \Delta \to \mathbb{R}$ is a signed measure which extends s. Then

$$m(A) + m(B^c) + m(C) + m(D^c) = 2m(X),$$

and, analogously,

$$m(A^c) + m(B) + m(C^c) + m(D) = 2m(X).$$

But if in the left-hand side we replace m by s, we obtain 0 in the first equality, and we obtain 4 in the second. A contradiction, thus, such an extension m of s does not exist.

It turns out, however, that many concrete logics do allow for the latter kind of extension. Such are for instance the logics of the following type (the conceptual value of these logics within various questions of quantum theories has been indicated in [4], [14], [17], etc.). Given a finite set X of even cardinality, we denote by X_{even} the concrete logic of all subsets of X whose cardinality is even.

Theorem 2.3 *Let X be a finite set of an even cardinality. Let s be a state on X_{even}. Then s can be extended as a signed measure over $\exp X$.*

PROOF: We can suppose that $\operatorname{card} X \geq 4$ (otherwise, the result is trivial). Take an arbitrary triple x, u, v of distinct elements of X. Let

$$\overline{m}(x) = \frac{1}{2}(s(\{x, u\}) + s(\{x, v\}) - s(\{u, v\})).$$

We claim that $\overline{m}(x)$ is independent of the choice of u, v. Further, we claim that letting x vary in X, we have defined a mapping $\overline{m} : X \to I\!\!R$ with the property that upon setting $m(A) = \sum_{a \in A} \overline{m}(a)$ for any subset A of X, this m constitutes a signed measure which extends s.

Let us first check that our definition of $\overline{m}$ is correct, i.e. let us show that $\overline{m}(x)$ does not depend upon the choice of u, v. Take first a couple v, w such that x, u, v, w are distinct. Let us show that both the couples u, v and v, w define the same $\overline{m}(x)$. Write

$$f(x, u, v) = s(\{x, u\}) + s(\{x, v\}) - s(\{u, v\}).$$

We want to show that $f(x, u, v) = f(x, v, w)$. Let us compute the difference

$$
\begin{aligned}
f(x, u, v) - f(x, v, w) &= s(\{x, u\}) + s(\{x, v\}) - s(\{u, v\}) \\
&\quad - s(\{x, v\}) - s(\{x, w\}) + s(\{v, w\}) \\
&= \big(s(\{x, u\}) + s(\{v, w\})\big) - \big(s(\{x, w\}) + s(\{u, v\})\big) \\
&= s(\{x, u, v, w\}) - s(\{x, u, v, w\}) = 0.
\end{aligned}
$$

Further, for a general element $t \in X$ distinct from x, u, v, w, we have $f(x, u, v) = f(x, v, w) = f(x, w, t)$, proving independence of the value $\overline{m}(x)$ on the choice of u and v. Let us check that m defined as above extends s. Take a set $A = \{x, y\} \subseteq X$. Then we can find u and v such that x, u, v, w are distinct and therefore

$$
m(A) = \overline{m}(x) + \overline{m}(y) = \frac{1}{2}\big(f(x, u, v) + f(y, u, v)\big)
$$

$$
= \frac{1}{2}\big(s(\{x, u\}) + s(\{x, v\}) - s(\{u, v\}) + s(\{y, u\}) + s(\{y, v\}) - s(\{u, v\})\big)
$$

$$
= \frac{1}{2}\big(2s(\{x, y, u, v\}) - 2s(\{u, v\})\big) = s(\{x, y\}).
$$

The proof is complete. □

Let us comment shortly on the previous result. Firstly, in the logic X_{even} we generally cannot extend states as states (see also [1, Th. 3.5.1(v)]).

Example 2.4 *Let $X = \{1, 2, \ldots, 2k\}$, $k \in \mathbb{N}$, $k \geq 2$. Consider the state s on X_{even} such that $s(\{1, c\}) = 0$ and $s(\{b, c\}) = \frac{1}{k-1}$ for all $b, c \in X \setminus \{1\}$, $b \neq c$. If m is any signed measure on $\exp X$ extending s we have $m(\{1\}) = -\frac{1}{2(k-1)}$ and $m(\{c\}) = \frac{1}{2(k-1)}$ for all $c \in X \setminus \{1\}$.*

Secondly, observing that each two-valued state on X_{even}, with card $X \geq 6$, has to be a Dirac state (i.e. a state sitting in a point), we see that there must be extreme states on X_{even} which are not two-valued (in fact, in [8] we have constructed some). And, thirdly, it is worthwhile observing that Th. 2.3 can also be proved, like many extension theorems which happen to hold, by the well-known criterion of Horn and Tarski (see e.g. [1, Def. 3.2.1, 3.2.2, and Th. 3.2.5, 3.2.10]). In the case of Th. 2.3 it would present another proof of a similar complexity. A minor advantage of the proof method used in Th. 2.3 is that it gives the result for group-valued measures for the groups which allow for dividing by 2. Observe in passing that, for instance, the extension problem of course-grained measures as treated in [5] and [10] finds in our opinion the Horn-Tarski criterion an effective proof device (see [3]).

The logics of X_{even} present a distinguished example of so-called difference-closed concrete logics. The latter logics were introduced in [10] under the name of symmetric concrete logics. In this note, let us use our notation (in our opinion more suggestive).

Definition 2.5 *A concrete logic Δ on a set X is said to be* difference-closed *if it is closed under the formation of symmetric differences, i.e., if for any couple of $A, B \in \Delta$ we have*

$$A \delta B = (A \setminus B) \cup (B \setminus A) \in \Delta \, .$$

It seems natural to conjecture that Th 2.3 can be generalized to all (finite) difference-closed logics. However, this is not the case.

Example 2.6 *Let $X = \{0, 1, 2, \ldots, 9\}$. Consider the smallest concrete logic L of subsets of X which is difference-closed and contains the sets $A = \{0, 1, 4, 7\}$, $B = \{0, 2, 5, 8\}$, $C = \{0, 1, 2, 3\}$ and $D = \{0, 4, 5, 6\}$. (For the reader acquainted with the theory of orthomodular lattices, the logic L is a representation of MO_{15} (thus, difference-closed logics can be lattices without being Boolean). The logic L has 15 mutually non-disjoint elements consisting of 4 points and 15 elements consisting of 6 points that constitute the complements of the former ones).*

Let s be a state on L such that $s(A) = s(C) = s(A\delta B) = 1$ and $s(B) = s(D) = s(C\delta D) = 0$ (there obviously are such states). Let us show that s cannot be extended as a signed state on $\exp X$.

Looking for a contradiction, suppose that m is such an extension. Since $A \cap B = C \cap D = \{0\}$, we obtain

$$2m(\{0\}) = s(A) + s(B) - s(A\delta B) = 0,$$

and, also,

$$2m(\{0\}) = s(C) + s(D) - s(C\delta D) = 1.$$

We have reached a contradiction, verifying the required property.

When we find ourselves within the area of difference-closed concrete logics, it seems of interest to deal with rather special states.

Definition 2.7 *Let L be a difference-closed concrete logic on a set X and let $s :$ $\Delta \to [0,1]$ be a state on Δ. We say that s is subadditive if $s(A\delta B) \leq s(A)+s(B)$ for any pair $A, B \in \Delta$.*

The following result indicates the meaning of subadditivity in our context.

Theorem 2.8 *Let X be a finite set and let $L = (X, \Delta)$ be a difference-closed concrete logic. Let s be a state on Δ which allows for an extension, m, over $\exp X$ as a signed measure. Let the intersections of pairs of sets of L generate all atoms of a Boolean algebra on X. Then s is subadditive if and only if m is a state.*

PROOF: Compute $s(A) + s(B) - s(A\delta B)$. We obtain

$$s(A) + s(B) - s(A\delta B)$$
$$= m(A \setminus B) + m(A \cap B) + m(B \setminus A) + m(A \cap B) - m(A \setminus B) - m(B \setminus A)$$
$$= 2m(A \cap B) .$$

We therefore see that $m(A \cap B)$ is nonnegative precisely when $s(A) + s(B) - s(A\delta B)$ is nonnegative, which occurs precisely when s is subadditive. We see that the extension m is a state on the Boolean algebra generated by Δ. By the classical theorem, m can be further extended over $\exp X$ as a measure. The proof is complete. □

Getting back to Th 2.3 interplayed with Th 2.8, we obtain the following corollary.

Theorem 2.9 *Let X be a set of even cardinality. Let s be a state on X_{even}. Then s can be extended over $\exp X$ as a state if and only if s is subadditive.*

A natural question arises if a difference-closed logic allows for extensions of states over $\exp X$ as signed measures. Regretfully, it is not the case. We demonstrate it in our final result.

Example 2.10 *Let us again consider the logic L of the previous Ex. 2.6. Let us consider the state s defined by setting*

$$s(A) = \frac{1}{3}, \qquad s(B) = \frac{1}{4}, \qquad s(C) = \frac{2}{5}, \qquad and$$

$$s(Y) = \frac{1}{2} \quad for \ any \quad Y \notin \{A, B, C, A^c, B^c, C^c\}.$$

It is not difficult to check that s is a subadditive state on L. This state s cannot be extended over $\exp X$ as a signed measure. Indeed, let m be such an extension. Then

$$2\,m(\{0\}) = m(A)+m(B)-m(A\delta B) = s(A)+s(B)-s(A\delta B) = \frac{1}{3}+\frac{1}{4}-\frac{1}{2} = \frac{1}{12},$$

whereas, analogously,

$$2\,m(\{0\}) = m(C) + m(D) - m(C\delta D) = s(C) + s(D) - s(C\delta D) = \frac{2}{5} + \frac{1}{2} - \frac{1}{2} = \frac{2}{5}\,.$$

This is a contradiction.

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